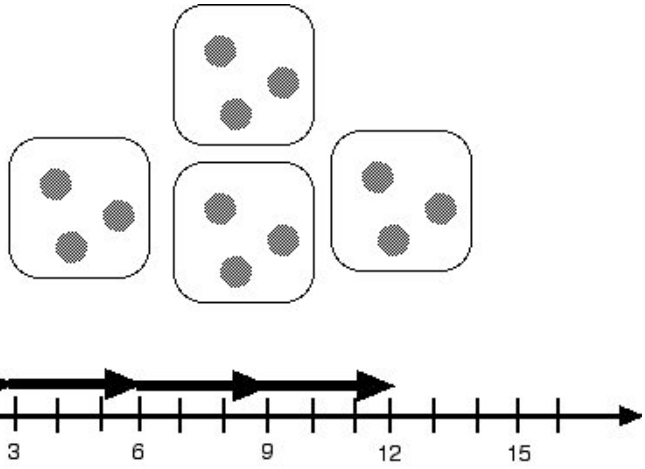
The Meanings of the Arithmetic Operations

The arithmetic operations are mathematical models (symbolic representations/notational systems/sign systems) of certain situations. A model is a way of representing some feature of reality. It is a way of looking at something. Models are not true or false. They are simply useful or not useful for some purpose. We tend to view the world through models in order to make sense of it. The models we use provide us with lenses through which human behaviour and other phenomena are organized, examined, and evaluated. For example, a set of moral principles is a model of how human beings are supposed to behave. There are numerous models that pertain to teaching. Piaget's model of child development is one example.

Consider the two situations shown in the diagram from the point of view of modeling them mathematically.



Each situation can be viewed as four groups with three in each group. This can be represented or modeled mathematically by the number sentence: '4 x 3 = 12'.

Note that, for 4 x 3 = 12, '12' is not the answer to anything because there is no question. The matter is not about questions and answers. It is about mathematically modeling or representing something. If one wishes to view the number sentence '4 x 3 = 12' from the point of view of answers, then there are three "answers". '4' is the answer to how many groups; '3' is the answer to how many objects in each group; and '12' is the answer to how many objects in all. Each number must be "figured out" before it can be placed in its appropriate place in the number sentence.

The four arithmetic operations can be organized into three themes:

- ★ **combining** (addition, multiplication)
- ★ **separating** (subtraction, division)
- ★ **comparing** (subtraction, multiplication - a Middle Years matter)

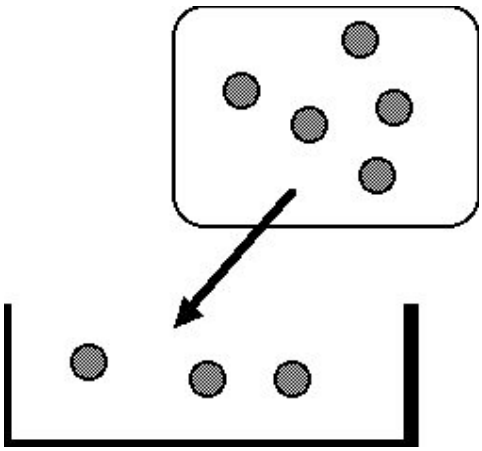
Each meaning has a TEMPLATE associated with it. This template is the mathematical way of representing the situation. Students are not expected to memorize the template, but to know it in their own way.

All the meanings relevant to the early years mathematics curriculum follow. These meanings are important to developing effective routine problem-solving skills.

COMBINING

(1) The 'put together' meaning of addition

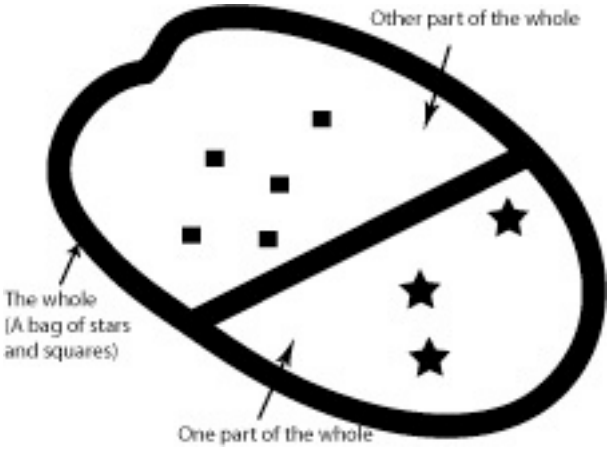
Addition has one meaning – put together. This meaning is represented by a number sentence such as '3 + 5 = 8'. In the diagram, the bottom container is intended to represent "all - the pot". Each symbol in the number sentence has a particular role. '3' is the count of what we already have in the "pot" . '+' indicates that we bring additional stuff into the "pot". '5' is the count of the additional stuff. '=' indicates an action of counting up what we now have in the "pot". '8' is the count of what is in the "pot" after the 'put together' action is over.



The 'put together' template: Stuff set #1 + stuff set #2 = all stuff

Another way to understand combining in the 'put together' sense is as a 'part/whole' situation. The action of 'putting together' is imagined as having been already done. The diagram on the right shows a 'part/whole' situation (a bag with squares and stars in it).

The number sentence: 5 + 3 = 8 would represent the situation. The whole (8 things) consists of 5 squares and 3 stars



(2) The 'groups of' meaning of multiplication

The 'groups of' meaning of multiplication concerns an existing arrangement of equal groups of objects. The multiplication symbol ('X') models the static situation that exists after equal groups have been formed rather than on the action of actually forming equal groups. The action of actually forming the groups is separating/splitting up and that is modeled by division.

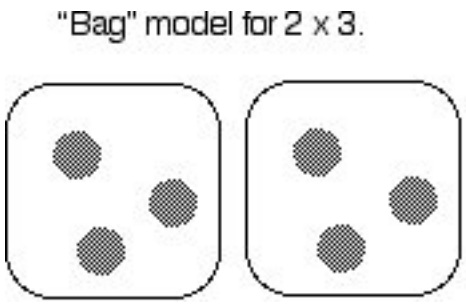
There is a convention attached to the 'groups of' meaning of multiplication.

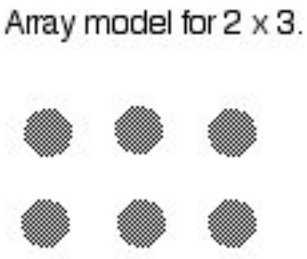
For example, in '5 x 3', the first number '5' represents the number of groups; the second number '3' represents the number in each group. Using this convention emphasizes the meaning of how many groups and how many in each group for a particular situation. Other conventions such as 'write the bigger number first' do not emphasize meaning.

The 'groups of' template: # of groups x number in group = all stuff

Two basic pictorial models are used for the 'groups' of meaning of multiplication: a "bag" model and an array model.

The diagrams illustrate a picture of 2 x 3 for each of these models. For the bag model, the first number indicates the number of bags (the group count).



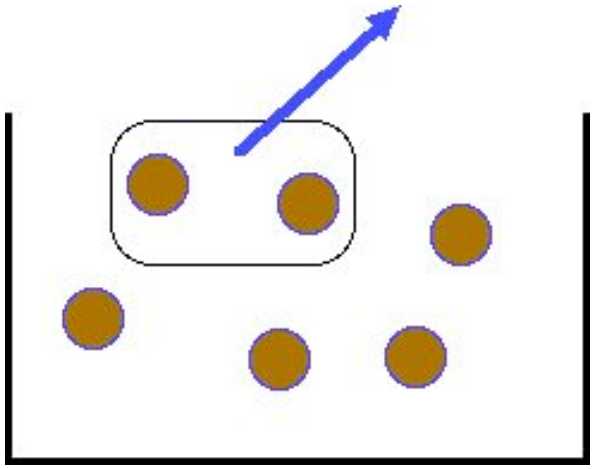


For the array model, the first number is the number of rows and the second number is the number of columns (like the convention used for a spreadsheet).

SEPARATING

(1) The 'take away' meaning of subtraction

The 'take away' meaning is the fundamental meaning of subtraction. It involves an action of removing things away from a "pot" and counting up what is left. Here is a picture of 6 – 2 = 4, from the point of view of the 'take away' meaning.



The 'take away' template: **All stuff – removed stuff = stuff left**

There are some points to keep in mind when teaching the 'take away' meaning of subtraction.

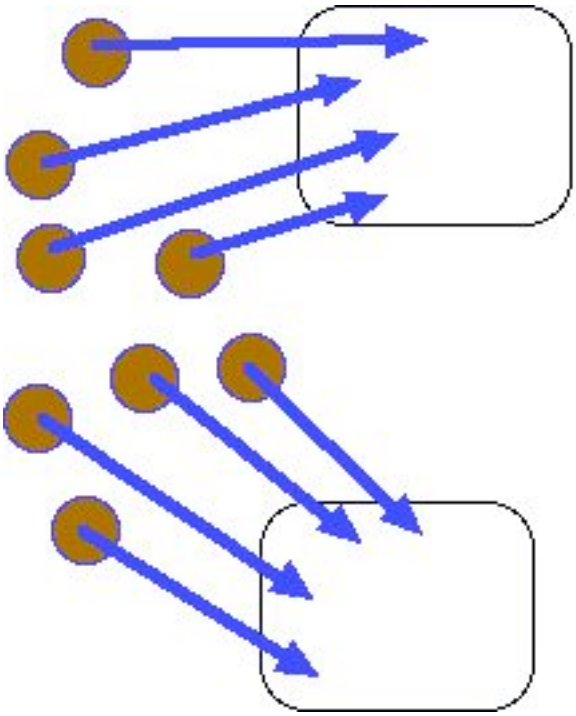
Language is important. It is better to use the generic term 'subtract' rather than the context-specific term 'take away' when referring to the arithmetic operation of subtraction.

However, it is appropriate to use the term 'take away' when referring to the actual action of taking away. This means that, for number sentences such as '7 - 3 = 4', it is preferable to say "7 subtract 3 is/equals 4" rather than "7 take away 3 is/equals 4". Using the term 'take away' to refer to the operation of subtraction may hinder the development of 'routine problem-solving skills because real world subtraction problems do not always involve the 'take away' meaning (sometimes they involve the comparison meaning of subtraction).

Another inadvisable word to use when referring to subtraction is 'minus'. Students are likely aware of that word in relation to temperature. By using the word 'minus' to mean 'subtract', you can create confusion for students now and in later years (when they learn about integers).

(2) The 'splitting up into equal groups' meaning of division

There is only one meaning of division. It represents an action that involves splitting up a collection of objects into equal groups. Here is a picture of 8 ÷ 4 = 2.



There are a couple of matters to keep in mind when teaching the meaning of division. Students have been interpreting a collection of equal groups in a multiplication way (the 'groups of' meaning). It may not be wise to use the same picture for illustrating multiplication and division. For multiplication, use static pictures. For division, use pictures that convey an action of splitting up into two equal groups (such as in the diagram here).

The 'splitting up' template: **All stuff ÷ number in group = # of groups**

There are two very different sorts of questions that one can ask when dividing.

One question concerns "How many groups can I form?" and the other concerns "How many are in each group?" In real world application, both questions are represented by the same number sentence, for example; '15 ÷ 3 = ?'. The '?' can represent either 'the number of groups' or 'the number in each group'. But which is it?

Pay attention to both types of questions when teaching students to solve routine problems involving division. One way to do that is to be STRICT when first teaching what division means and simple problems involving division. Being strict means paying "religious" attention to the template: all ÷ # in each group = # of groups. Place the '?' in its proper slot in the template. This forces students to pay attention to meaning, rather than to other things like number size.

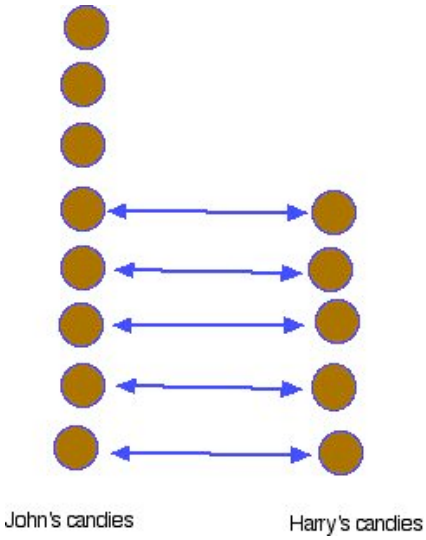
COMPARING

Comparing involves making decisions about more/less/equal, shorter/taller/same as etc. Comparison is modeled by subtraction and multiplication. We will consider subtraction here.

The comparison meaning of subtraction

The comparison meaning of subtraction does not involve an action of removing or taking away. It involves comparing two sets as to more/less/same. The heart of the matter for comparing by subtraction is matching the objects of two sets and counting what is unmatched. The unmatched stuff is typically referred to as the difference between the two sets.

Consider the following. John has 8 candies. Harry has 5 candies. If the candies of the two boys are matched, we will find that there will be 3 candies unmatched. We refer to this situation in many ways: "The difference between two sets of candies is 3." or "John has 3 more candies than Harry." or "Harry has 3 fewer candies than John." In mathematics we can write this as '8 - 5 = 3', a number sentence that looks identical to one that might involve the 'take away' meaning of subtraction. Notice that there is no removal of stuff when John's and Harry's candies are being compared. John still has 8 candies and Harry still has 5. In fact, there are 13 candies in all involved in the situation. Here is a picture of the candies and the matching process.



The 'comparison' template: stuff set A - Stuff set B = Unmatched stuff

There are some matters to keep in mind when teaching the comparison meaning of subtraction.

If students have been taught to say 'take away' when referring to the subtraction symbol, '-', there are likely to be difficulties. They may be confused and even trapped by the faulty language. That is why the generic language, 'subtract', is better language for the mathematical symbol, '-'. The actions/situations can be described according to what is happening (losing, comparing, etc.).

Attention needs to be paid to the use of appropriate language such as 'more/less/fewer', 'taller than/shorter than'. In relation to this, we subtract the smaller number from the larger number when using the comparison meaning of subtraction. The mathematical idea of positive/negative is expressed by language such as bigger/smaller. This way negative numbers are not needed.